HOMOGENEOUS HELICAL MOTION IN A CONE

(ODNORODNOE VINTOVOE DVIZHENIE V KONUSE)

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The problem of homogeneous helical motion in a cone of finite dimensions is examined (the similar problem for an infinite cone has been formulated by Vasil'ev [1]).

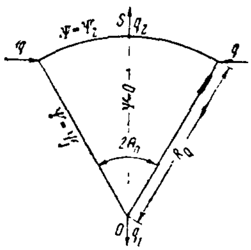

Fig. 1.

We shall consider that the fluid is ideal and incompressible and that within a cone there is a homogeneous helical flow symmetric with respect to the axis of the cone. Fluid enters the cone through an annular slit in the amount q units per second and leaves through the apex and the point S in the amounts q_1 and q_2, respectively.

The length of a generator of the cone is R_0 and the half-angle of the apex is θ_0. From above the cone is bounded by a spherical surface of radius R_0, which has an aperture S on the axis of the cone through which the fluid flows out (Fig. 1). Such a scheme reflects approximately the principle of action of a hydro-cyclone.

The problem is reduced to solving the inhomogeneous differential equation [1]

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+k^2\psi=-kC \qquad (k,\ C=\mathrm{const}) \tag{1}$$

in the region $0\leqslant r\leqslant R_0$, $0\leqslant\theta\leqslant\theta_0$ for the boundary conditions

$$\psi(r,\ 0)=0,\quad \psi(r,\ \theta_0)=\psi_1,\quad \psi(R_0,\ 0)=\psi_2\quad \left(\psi_1=-\frac{q_1}{2\pi},\ \psi_2=+\frac{q_2}{2\pi}\right) \tag{2}$$

In place of the stream function $\psi(r,\ \theta)$ we shall introduce a new function $u(r,\ \theta)$ which is connected with $\psi(r,\ \theta)$ by the relation

$$u(r,\ \theta)=\psi(r,\ \theta)-\psi_1\frac{\sin^2\theta}{\sin^2\theta_0} \tag{3}$$

Equation (1) takes the form

$$\frac{\partial^2 u}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\theta}\right)+k^2u=-kC-\frac{\psi_1}{\sin^2\theta_0}\left(k^2-\frac{2}{r^2}\right)\sin^2\theta \tag{4}$$

The boundary conditions for the function $u(r, \theta)$ take the form

$$u(r, 0)=0, \qquad u(r, \theta_0)=0, \qquad u(R_0, \theta)=\psi_2-\psi_1\frac{\sin^2\theta}{\sin^2\theta_0} \tag{5}$$

We shall seek the function $u(r, \theta)$ in the form of the series

$$u(r, \theta)=\sum_{n=1}^{\infty} M_n(r)N_n(\theta) \tag{6}$$

ordered with respect to the eigen-functions $N_n(\theta)$. We shall determine the form of the eigen-functions. Separating the variables in the homogeneous equation which corresponds to the inhomogeneous equation (4), we obtain for the determination of the function $N(\theta)$ the differential equation

$$\sin\theta\left(\frac{1}{\sin\theta}N'\right)'+\nu(\nu+1)N=0 \tag{7}$$

which by the substitution $x = \cos\theta$ reduces to the form

$$\frac{d^2N}{dx^2}+\frac{\nu(\nu+1)}{1-x^2}N=0$$

The general integral of this equation for ν not equal to zero or a negative integer is written as

$$N_\nu(x)=\sqrt{1-x^2}\,[A_\nu P_\nu^1(x)+B_\nu Q_\nu^1(x)]$$

or, transforming to the variable θ

$$N_\nu(\theta)=\sin\theta\,[A_\nu P_\nu^1(\cos\theta)+B_\nu Q_\nu^1(\cos\theta)]$$

where $P_\nu^1(\cos\theta)$ and $Q_\nu^1(\cos\theta)$ are associated Legendre functions.

Since the function $u(r, \theta)$ vanishes for $\theta = 0$ and the product $Q_\nu^1(\cos\theta)\sin\theta$ tends to a constant as $\theta \to 0$, B_ν must be set equal to zero to satisfy the boundary condition on the axis.

To satisfy the boundary condition on the lateral surface of the cone it is necessary to satisfy the condition $P_\nu^1(\cos\theta_0) = 0$; the eigenvalues ν_n are determined from this transcendental equation.

The functions

$$N_n(\theta)=\sin\theta P_{\nu_n}^1(\cos\theta)$$

are the eigen-functions of the boundary-value problem under consideration.

Multiplying both sides of (6) by $\rho(\theta)\ N_n(\theta)\ d\theta$, where $\rho(\theta)$ is a weighting function, and integrating between the limits from 0 to θ_0, we obtain

$$M_n(r)=\frac{1}{N_n^2}\int_0^{\theta_0}u(r,\ \theta)\,\rho(\theta)\,N_n(\theta)\,d\theta \qquad \left(N_n^2=\int_0^{\theta_0}\rho(\theta)\,[N_n(\theta)]^2d\theta\right) \tag{8}$$

where N_n^2 is the norm of the eigen-functions. From Equation (7) it is seen that $\rho(\theta) = \cos\theta$.

To determine the coefficients of the series $M_n(r)$ we shall multiply Equation (4) by

$$N_n^{-2}\,\rho(\theta)\sin\theta P_{\nu_n}{}^1(\cos\theta)d\theta$$

and integrate from 0 to θ_0; using (8) and the first two conditions of (5) we obtain

$$\frac{d^2M_n}{dr^2}+\left[k^2-\frac{\nu_n(\nu_n+1)}{r^2}\right]M_n=-(kC\alpha_n+k^2\psi_1\beta_n)+\psi_1\beta_n\frac{2}{r^2} \tag{9}$$

Here and below, the designations

$$\alpha_n=\frac{1}{N_n^2}\int_0^{\theta_0}P_{\nu_n}'(\cos\theta)\,d\theta, \qquad \beta_n=\frac{1}{\sin^2\theta_0\,N_n^2}\int_0^{\theta_0}\sin^2\theta P_{\nu_n}{}^1(\cos\theta)\,d\theta \tag{10}$$

have been introduced.

The function $M_n(r)$ is subject to the boundary conditions:

1) for $r \to 0$, the function $M_n(r)$ must have a finite value;

2) for $r = R_0$, in accordance with (8), (10) and the third condition of (5)

$$M_n(R_0)=\alpha_n\psi_2-\beta_n\psi_1 \tag{11}$$

By substituting $M_n(r) = \sqrt{r}\ L_n(r)$ and $z = kr$, Equation (9) reduces to the form

$$\frac{d^2L_n}{dz^2}+\frac{1}{z}\frac{dL_n}{dz}+\left[1-\frac{(\nu_n+{}^1/_2)^2}{z^2}\right]L_n=-\left(\frac{C\alpha_n}{k}+\psi_1\beta_n\right)k^{1/2}z^{-1/2}+2\psi_1k^{1/2}\beta_nz^{-5/2}$$

The general solution of this equation is written as [2]

$$L_n(z) = A_n J_{\mu_n}(z) + B_n J_{-\mu_n}(z) - \left(\frac{C\alpha_n}{k} + \psi_1\beta_n\right) k^{1/2} s_{1/2,\,\mu_n}(z) + 2\psi_1\beta_n k^{1/2} s_{-3/2,\,\mu_n}(z) \quad (12)$$

where $S_{\mu,\nu}(z)$ is the Lommel function; for brevity, the designation has been introduced

$$s_{\mu,\,\nu}(z) = \frac{z^{\mu+1}}{(\mu+1)^2 - \nu^2} - \frac{z^{\mu+3}}{[(\mu+1)^2 - \nu^2]\,[(\mu+3)^2 - \nu^2]} + \dots$$

$$= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)} \left(\frac{1}{2}z\right)^{2m+2}$$

We shall transform (12) back to the function $M_n(z)$ and to the variable r:

$$M_n(r) = A_n \sqrt{r}\, J_{\mu_n}(kr) + B_n \sqrt{r}\, J_{-\mu_n}(kr) - \\ - \left(\frac{C\alpha_n}{k} + \psi_1\beta_n\right)\sqrt{kr}\, s_{1/2,\,\mu_n}(kr) + 2\psi_1\beta_n \sqrt{kr}\, s_{-3/2,\,\mu_n}(kr)$$

For $\nu_n > 0$ the function

$$\sqrt{r}\, J_{-\mu_n}(kr) \to -\infty, \text{ for } r \to 0$$

Therefore B_n must be set equal to zero. Using condition (11) we determine the constant of integration

$$A_n = \frac{1}{\sqrt{R_0}\, J_{\mu_n}(kR_0)} \left[\alpha_n\psi_2 - \beta_n\psi_1 + \left(\frac{C\alpha_n}{k} + \beta_n\psi_1\right)\sqrt{kR_0}\, s_{1/2,\,\mu_n}(kR_0) - \\ - 2\psi_1\beta_n \sqrt{kR_0}\, s_{-3/2,\,\mu_n}(kR_0)\right]$$

The final expression for the stream function can now be written

$$\psi(r,\,\theta) = \varphi_1 \frac{\sin^2\theta}{\sin^2\theta_0} + \\ + \psi_1 \sum_{n=1}^{\infty} \left\{\left[\alpha_n \frac{\psi_2}{\psi_1} - \beta_n \left(1 + 2\sqrt{kR_0}\, s_{-3/2,\,\mu_n}(kR_0) - \sqrt{kR_0}\, s_{1/2,\,\mu_n}(kR_0)\right)\right]\sqrt{\frac{r}{R_0}}\, \frac{J_{\mu_n}(kr)}{J_{\mu_n}(kR_0)} + \\ + \beta_n \sqrt{kr}\left(2 s_{-3/2,\,\mu_n}(kr) - s_{1/2,\,\mu_n}(kr)\right)\right\} \sin\theta\, P_{\nu_n}{}^1(\cos\theta) + \\ + \frac{C}{k} \sum_{n=1}^{\infty} \alpha_n \left[\sqrt{kR_0}\, s_{1/2,\,\mu_n}(kR_0) \sqrt{\frac{r}{R_0}}\, \frac{J_{\mu_n}(kr)}{J_{\mu_n}(kR_0)} - \sqrt{kR}\, s_{1/2,\,\mu_n}(kr)\right] \sin\theta\, P_{\nu_n}{}^1(\cos\theta) \quad (13)$$

The velocity components are expressed in terms of $\psi(r, \theta)$ by the formulas

$$v_\varphi = \frac{C + k\psi}{r \sin\theta}, \qquad v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

From the first formula it is seen that the circumferential velocity becomes infinite on the axis of the cone for $C \neq 0$. This explains the formation of the air column at the axis of the cone which, in fact, is observed in the hydro-cyclone.

Introducing the dimensionless quantities $\rho = r/R_0$, $k = kR_0$, $\gamma = \psi_2/\psi_1$, Formula (13) can be rewritten in a form more convenient for computations:

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta}{\sin^2\theta_0} + \sum_{n=1}^{\infty}\Big\{[\alpha_n\gamma - \beta_n(1 + 2\sqrt{\varkappa}s_{-3/2,\mu_n}(\varkappa) - \sqrt{\varkappa}s_{1/2,\mu_n}(\varkappa))]\sqrt{\rho}\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} +$$

$$+ \beta_n\sqrt{\varkappa\rho}\,(2s_{-3/2,\mu_n}(\varkappa\rho) - s_{1/2,\mu_n}(\varkappa\rho))\Big\}\sin\theta P_{\nu_n}^{1}(\cos\theta) +$$

$$+ \frac{c}{k\psi_1}\sum_{n=1}^{\infty}\alpha_n\left[\sqrt{\varkappa}s_{1/2,\mu_n}(\varkappa)\sqrt{\rho}\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho}s_{1/2,\mu_n}(\varkappa\rho)\right]\sin\theta P_{\nu_n}^{1}(\cos\theta) \qquad (14)$$

If we pass to the limit as $k \to 0$ in the solution thus found, we obtain the solution to the problem of the potential motion of a fluid in a cone. To obtain such a solution k could have been set equal to zero in (9) and the function $M_n(r)$ then determined from the differential equation so obtained. Both paths lead to one and the same result, namely

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta}{\sin^2\theta_0} + \sum_{n=1}^{\infty}\left\{\left[\alpha_n\gamma - \beta_n + \frac{2\beta_n}{\nu_n(\nu_n+1)}\right]\rho^{\nu_n+1} - \frac{2\beta_n}{\nu_n(\nu_n+1)}\right\}\sin\theta P_{\nu_n}^{1}(\cos\theta). \qquad (15)$$

Solutions to problems in other special cases can also be obtained from (14). Thus, $\gamma = 0$ corresponds to the absence of the aperture at the point S, and $\gamma = 1$ corresponds to the absence of the annular slit. A method for determining the eigen-values ν_n and for calculating N_n^2, α_n, β_n remains to be shown.

The formula

$$P_\nu^{m}(\cos\theta) = (-1)^m\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}P_\nu^{-m}(\cos\theta)$$

is valid for integral values of m.

From this formula it is seen that the functions $P_\nu^{1}(\cos\theta)$ and

$P_\nu^{-1}(\cos\theta)$ have the same roots ν_n. A method for determining the roots of the equation $P_\nu^{-1}(\cos\theta_0) = 0$ has been described in the work of MacDonald [3].

For small values of θ_0 the following approximate formula can be used to determine the roots of the transcendental equation $P_\nu^{-1}(\cos\theta_0) = 0$

$$\nu_n + \frac{1}{2} = \frac{x_n}{2\sin{}^1/_2\,\theta_0}\left[1 - \frac{1}{6}\sin^2\frac{\theta_0}{2}\left(1 - \frac{3}{x_n^2}\right)\right]$$

where x_n is the nth root, different from zero, of the equation $J_1(x) = 0$.

But if θ_0 is not small, then the formula [3]

$$\nu_n + \frac{1}{2} = x_n + \frac{b_1}{\theta_0(1+n_n)} + \frac{b_2}{\theta_0(1+x_n)(2+x_n)} - \frac{a_1 b_1}{\theta_0(1+x_n)^2} + \ldots \quad (16)$$

can be used for the same purpose, where

$$x_n = \frac{\pi}{2\theta_0}\left(2n + m + \frac{3}{2}\right)$$

$$a_1 = \frac{1^2 - 4m^2}{2^2}\,\frac{\cos({}^1/_2\pi - \theta_0)}{2\sin\theta_0}, \qquad b_1 = \frac{1^2 - 4m^2}{2^2}\,\frac{\sin({}^1/_2\pi - \theta_0)}{2\sin\theta_0}$$

$$a_2 = \frac{(1^2 - 4m^2)(3^2 - 4m^2)}{2^4(2\sin\theta_0)^2\,2!}\cos(\pi - 2\theta_0), \qquad b_2 = \frac{(1^2 - 4m^2)(3^2 - 4m^2)}{2^4(2\sin\theta_0)^2\,2!}\sin(\pi - 2\theta_0), \ldots$$

In the case under consideration $m = 1$. By means of simple calculations we find that

$$N_n^2 = -\frac{\sin\theta_0 P_{\nu_n}(\cos\theta_0)}{2\nu_n + 1}\,\nu_n(\nu_n + 1)\,\frac{\partial P_{\nu_n}^1(\cos\theta_0)}{\partial\nu_n} \quad (17)$$

$$\alpha_n = \frac{P_{\nu_n}(\cos\theta_0) - 1}{N_n^2}, \qquad \beta_n = \frac{(\nu_n + 1)\,P_{\nu_n - 1}^1(\cos\theta_0)}{\sin\theta_0(\nu_n + 2)(\nu_n - 1)N_n^2}.$$

To calculate the derivative $\partial P_{\nu n}^1(\cos\theta_0)/\partial\nu_n$ we shall examine the identity

$$P_{\nu_n}^1(\cos\theta) = 0$$

Considering the left-hand side as a function of θ and ν_n, we find the complete differential

$$\frac{\partial P_{\nu_n}^1(\cos\theta)}{\partial\nu_n}\,d\nu_n + \frac{\partial P_{\nu_n}^1(\cos\theta)}{\partial\theta}\,d\theta = 0$$

Hence we have

$$\frac{\partial P_{\nu_n}{}^1(\cos\theta_0)}{\partial \nu_n} = -\frac{\partial P_{\nu_n}{}^1(\cos\theta)}{\partial\theta}\,\frac{1}{d\nu_n/d\theta}\bigg|_{\theta=\theta_0}$$

Using the recurrence relation

$$\sin\theta\,\frac{dP_{\nu_n}{}^1(\cos\theta)}{d\theta} = -\nu_n(\nu_n+1)\sin\theta P_{\nu_n}(\cos\theta) - \cos\theta P_{\nu_n}{}^1(\cos\theta)$$

and taking into consideration that $P_{\nu_n}{}^1(\cos\theta_0) = 0$, we obtain

$$\frac{dP_{\nu_n}{}^1(\cos\theta)}{d\theta}\bigg|_{\theta=\theta_0} = -\nu_n(\nu_n+1)P_{\nu_n}(\cos\theta_0)$$

Formula (16) gives ν_n as a function of θ; for this it is necessary to replace θ_0 in it by θ and to take into consideration that x_n, a_1, b_1, ... are, in turn, dependent on θ. Now, by virtue of (17)

$$N_n{}^2 = -\sin\theta_0\,\frac{\nu_n^2(\nu_n+1)^2}{2\nu_n+1}[P_{\nu_n}(\cos\theta_0)]^2\,\frac{1}{d\nu_n/d\theta}\bigg|_{\theta=\theta_0} \tag{18}$$

Associated Legendre functions with integral indices only have been tabulated. But in the problem under consideration the lower index of these functions may have any value. It is not feasible to calculate these functions by means of the hypergeometric series in which they are expressed because of the poor convergence of these series. For non-integral values of ν_n the function $P_{\nu_n}{}^1(\cos\theta)$ can be calculated by interpolation, for example using parabolas.

The computations are considerably simplified by making use of the asymptotic representations of the functions $P_{\nu_n}{}^1(\cos\theta)$ and $P_{\nu_n}(\cos\theta)$ for large values of ν_n. In [4] the asymptotic formulas

$$P_l{}^m(\cos\theta) = \left(l+\frac{1}{2}\right)^m\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_{-m}\left[\left(l+\frac{1}{2}\right)\theta\right]$$

$$P_l(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\left[\left(l+\frac{1}{2}\right)\theta\right]$$

have been determined.

Good results are obtained even for $l = 10$. For $l = \nu_n$ and $m = 1$ we have

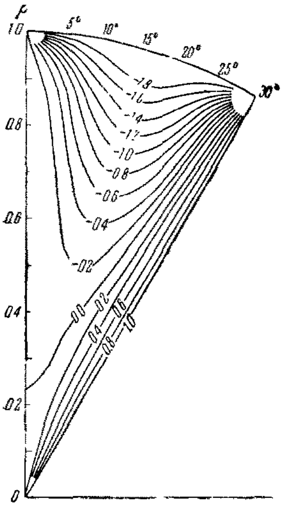

Fig. 2.

$$P_{\nu_n}{}^1(\cos\theta) = -\mu_n\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_1(\mu_n\theta)$$
$$P_{\nu_n}(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0(\mu_n\theta) \tag{19}$$

From the first formula it is seen that ν_n can be determined from the simple relation

$$\left(\nu_n + \frac{1}{2}\right)\theta_0 = x_n \tag{20}$$

where x_n is the nth root, different from zero, of the equation $J_1(x) = 0$.

If Expression (19) for $P_{\nu_n}{}^1(\cos\theta)$ is used we obtain for $N_n{}^2$

$$N_n{}^2 = \mu_n{}^2\int_\theta^{\theta_0} \theta J_1{}^2(\mu_n\theta)\,d\theta = \mu_n{}^2\frac{\theta_0{}^2}{2}J_0^2(\mu_n\theta_0)$$

For large arguments the asymptotic formula

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$$

is valid.

Taking this expression into consideration and also the fact that the function $J_0(\mu_n\theta_0)$ attains extrema at the points $\mu_n\theta_0$. we obtain the very simple formula

$$N_n{}^2 = \mu_n\frac{\theta_0}{\pi} \tag{21}$$

In calculating the quantity $N_n{}^2$ for the first root the relative error between Formulas (18) and (21) for the largest angle of practical interest $\theta_0 = 1/6\,\pi$ does not exceed 3%. With increasing root number this error decreases; it also decreases with decreasing angle θ_0.

The difference in the values of ν_n, calculated according to Formulas (16) and (20) for $\theta_0 = 1/6\,\pi$, appears in the third decimal point. All of this confirms the fact that Formulas (19) to (21) can be used for the calculations. Expression (14) for the stream function can be simplified with the aid of Formulas (19) and (21) to

$$\frac{\psi(\rho,\theta)}{\psi_1} = \frac{\sin^2\theta}{\sin^2\theta_0} - \frac{\pi}{\theta_0}\sum_{n=1}^{\infty}\left\{[a_n\lambda - b_n(1 + 2\sqrt{\varkappa}s_{-3/2,\,\mu_n}(\varkappa) - \sqrt{\varkappa}s_{1/2,\,\mu_n}(\varkappa))]\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} + \right.$$

$$+ b_n \sqrt{\varkappa\rho}\,(2s_{-3/2,\,\mu_n}(\varkappa\rho) - s_{1/2,\,\mu_n}(\varkappa\rho))\Big\} \sqrt{\theta\sin\theta} J_1(\mu_n\theta) -$$

$$- \frac{C}{k\psi_1}\frac{\pi}{\theta_0}\sum_{n=1}^{\infty} a_n \left[\sqrt{\varkappa} s_{1/2,\,\mu_n}(\varkappa)\sqrt{\rho}\,\frac{J_{\mu_n}(\varkappa\rho)}{J_{\mu_n}(\varkappa)} - \sqrt{\varkappa\rho}\, s_{1/2,\,\mu_n}(\varkappa\rho)\right] \sqrt{\theta\sin\theta} J_1(\mu_n\theta) \qquad (22)$$

Here

$$a_n = \left(\frac{\theta_0}{\sin\theta_0}\right)^{1/2} J_0(\mu_n\theta_0) - 1$$

$$b_n = -\frac{(\nu_n+1)(\nu_n - 1/2)}{(\nu_n+2)(\nu_n-1)\sin\theta_0}\left(\frac{\theta_0}{\sin\theta_0}\right)^{1/2} J_1\left[\left(\nu_n - \frac{1}{2}\right)\theta_0\right]$$

With the help of these formulas calculations have been carried out for the following initial data: $\theta_0 = 1/6\,\pi$, $\kappa = 4$, $C/k\psi_1 = -4$, $\gamma = -2$. Streamlines (Fig. 2) have been constructed on the basis of the results obtained. From the figure it is seen that the stream surfaces obtained correctly reflect the picture of the motion in a part of the fluid in a hydro-cyclone.

With increasing absolute value of the parameter $C/k\psi_1$ the separation line is lowered and closed streamlines appear near the cover, forming a stagnant zone. In this zone the fluid circulates without issuing from the cone.

It remains to note that the series (22) converges rapidly up to approximately $\rho = 0.9$, and that for $\rho > 0.9$ the convergence is weak and is weaker as the cover is approached.

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